# Schmidt number of pure continuous-variable bi-partite entangled states and methods of its calculation

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## Abstract

An entanglement measure for pure-state continuous-variable bi-partite problem, the Schmidt number, is analytically calculated for one simple model of atom-field scattering.

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## 1 Introduction

Quantum entanglement — non-classical correlations between parts of compound quantum system is one the most important topics of contemporary quantum theory, including quantum optics and quantum information [1].

One of the methods for analysis of bi-partite pure entangled states is the Schmidt decomposition [2–4] — representation of the given state as a sum of product terms, where the basic states are eigenvectors of marginal density matrices. For continuous-variable (CV) problems, the Schmidt decomposition gives an effectively finite-dimensional Hilbert space and entanglement is characterized by the Schmidt number — reciprocal of the sum of marginal density operator eigenvalues.

In a recent series of publications, see further references in [4,5], concerning the Schmidt decomposition-based analysis of various problems, the entanglement parameters were calculated mainly from numerical decomposition. In fact, the only exception is the case of double-Gaussian wave function with analytically known decomposition, see for example [6–9] or Gaussian mixed states [10].

The aim of the present paper is to discuss another way of the Schmidt number calculation — from marginal purity. This method is rarely mentioned in the published works, however in many cases it enables to obtain the Schmidt number (in exact or approximate form) analytically, and, for certain tasks, eliminate the need to calculate the decomposition itself.

## 2 CV pure-state entanglement and the Schmidt decomposition

The Schmidt decomposition, known also by different other names in various scientific fields [11], in quantum theory is an analysis of non-separable wavefunctions via representation

$$|\varphi^{(AB)}\rangle = \sum\nolimits_i \sqrt{\lambda}_i \, |\varphi_i^{(A)}\rangle \, |\varphi_i^{(B)}\rangle, \tag{1}$$

where the state vectors $|\varphi_i^{(A)}\rangle$ and $|\varphi_i^{(B)}\rangle$ forms orthonormal systems belonging to different parts of the composite quantum system and all $\lambda_i \geqslant 0$. These state vectors are eigenvectors of marginal density matrices of sub-systems, and coefficients $\lambda_i$ are corresponding eigenvalues. According to the decomposition (1), different Schmidt modes appears in parts of a system in pairs, reflecting correlations between sub-systems.

For CV systems, there is in principle infinite countable set of eigenvalues $\lambda_i$. However, for all quantum systems, sum of all eigenvalues is essentially finite, therefore only the limited number of eigenvalues has significant value, and the whole entangled system becomes effectively finite-dimensional [7].

The number of significant terms in representation (1) can be characterized by the value

$$K = 1/\sum\nolimits_i \lambda_i^2, \tag{2}$$

effective number of the Schmidt modes (or the Schmidt number). On the other hand, the Schmidt number is expressed via trace of squared marginal density matrix ($\mathrm{Tr}_A$ and $\mathrm{Tr}_B$ are partial traces over respective sub-system)

$$K = 1/\mathrm{Tr}\left(\rho_A^2\right) = 1/\mathrm{Tr}\left(\rho_B^2\right), \tag{3}$$

$$\begin{aligned} \rho_A &= \mathrm{Tr}_B\left(|\varphi^{(AB)}\rangle\langle\varphi^{(AB)}|\right), \\ \rho_B &= \mathrm{Tr}_A\left(|\varphi^{(AB)}\rangle\langle\varphi^{(AB)}|\right). \end{aligned} \tag{4}$$

Eigenvector decomposition is optimal from several points of view, see detail in [12], but it can be calculated analytically only in several special cases. Despite the mathematical methods to treat this problem

effectively [13], it is desirable to deal at least with entanglement degree analytically.

Schmidt number is closely related to "generalized entropies" (or generalized purities), proposed by Rényi, Tsallis [14] and others, see further references in [10, 15, 16]:

$$S_p = \frac{1 - \mathrm{Tr}\rho^p}{p-1}, \quad S_p^R = \frac{\ln\, \mathrm{Tr}\rho^p}{p-1}, \quad p > 1 \qquad (5)$$

($\rho$ is a marginal density matrix). Here the limit $p \to 1$ leads to usual Shannon-von Neumann entropy, standard parameter for entanglement charactetization [17] and the case $p = 2$ (purity, or "linear entropy") gives the Schmidt number

$$S_2 \equiv S_L = 1 - 1/K.$$

The Schmidt number has a special meaning. First of all, it can be measured in a single-photon counting experiment, see [7, 10, 18] and the references therein. From theoretical point of view, for the whole family of parameters (5), the Schmidt number is most easy to calculate, just using the formula (3), without knowledge of the decomposition eigenvalues.

# 3 Atom-photon scattering: a model

An entangled atom-field wavefunction in dimensionless momentum representation in simplest case depends on only one parameter [19–22]

$$C(k, q) = \frac{N\, \exp(-\delta q^2/\eta^2)}{\delta k + \delta q + i}, \quad \eta = \frac{\hbar\omega_0\, \sigma}{Mc\gamma} \qquad (6)$$

where $\eta$ — the control parameter, ratio of thermal (motional) line broadening $\hbar\omega_0\,\sigma/(Mc)$ to a natural linewidth $\gamma$, $N$ is a normalization constant.

For this model, in the Raman scattering regime [21], control parameter value can be as large as $\eta_R \approx 4500$, opening a way of experimental realization of high-entanglement regime.

In the papers [19–22], numerical treatment is applied for the Schmidt decomposition, leading to following expression for $K(\eta)$ at large $\eta$ (result of numerical fitting)

$$K = 1 + 0.28\ (\eta - 1)\,. \qquad (7)$$

On the other hand, a straightforward analytical integration according to (4) gives marginal (atom) density function

$$\rho(q_1, q_2) = \frac{\pi\, N^2\, \exp(-\delta q_1^2/\eta^2 - \delta q_2^2/\eta^2)}{(\delta q_1 - \delta q_2)/2i + 1}, \qquad (8)$$

together with a normalization constant $N^2 = \sqrt{2}/(\pi^{3/2}\,\eta)$.

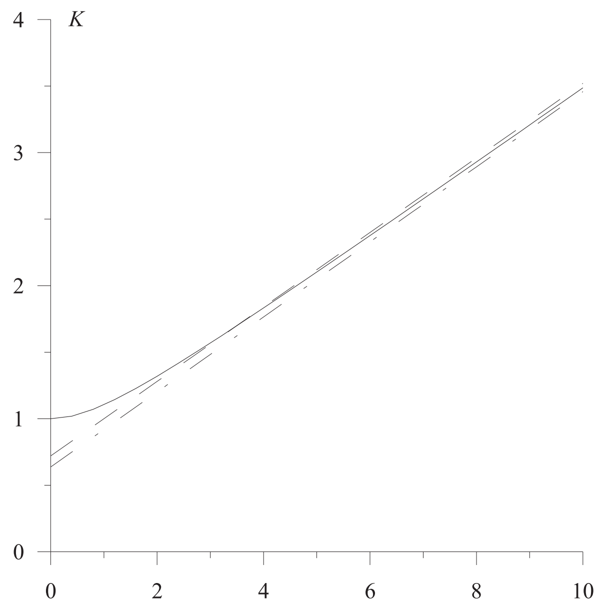


Figure 1: Different methods of the Schmidt number calculation. Solid line — exact dependence (9), dash-dot — asymptotic (10), dashed — Eberly's approximation (7).

Another step of integration according to (3) leads to analytical formula for the Schmidt number

$$K = \frac{\eta}{2\sqrt{\pi}} \frac{\exp(-4/\eta^2)}{1 - \mathrm{erf}(2/\eta)}, \qquad (9)$$

which has simple asymptotic form for $\eta \gg 1$

$$K = \frac{2}{\pi} + \frac{\eta}{2\sqrt{\pi}}. \qquad (10)$$

Resulting dependencies $K(\eta)$ for these three expressions are presented in Figure 3. It is seen, that all the graphs are quite close to each other.

# 4 Conclusion

The main task of the present paper — to provide a simplest example of an already known, but not widely used method of the Schmidt number calculation. The Schmidt decomposition and the Schmidt number prove to be quite efficient method for characterization of pure state bi-partite entanglement, however, its analytical calculation is possible just in a very limited number of cases. Among the used simple example, analytical (or semi-analytical) expressions can be found for another cases, for example, for more generalized two-parametric atom-field entanglement model [23].


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